Generation of all rational numbers in $(0,1)$.
https://www.linkedin.com/feed/update/urn:li:activity:6949237860680048640?utm_source=linkedin_share&utm_medium=member_desktop_web
Functions $f(x) = \frac{1}{1+x}, g(x) = \frac{x}{1+x}$ are defined on $(0,1)$. Is it possible starting from $\frac{1}{2}$ and using only operations $f(x), g(x)$ to obtain any rational number in $(0,1)$?.
Solution by Arkady Alt, **San Jose**, **California**, **USA.**
1. Let $g_n(x)$ be defined recursively by $g_0(x) := x$ and $g_{n+1}(x) = g(g_n(x)), n \in \mathbb{N} \cup \{0\}$.
Then $g_1(x) = g(x), g_2(x) = \frac{g(x)}{1+g(x)} = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}} = \frac{x}{1+2x}$ and for any $n \in \mathbb{N}$,
assuming $g_n(x) = \frac{x}{1+nx}$ we obtain $g_{n+1}(x) = \frac{\frac{x}{1+nx}}{1+\frac{x}{1+nx}} = \frac{x}{1+(n+1)x}$.
Thus, by MI proved that $g_n(x) = \frac{x}{1+nx}, n \in \mathbb{N} \cup \{0\}$.
Hence, $g_n(f(x)) = \frac{f(x)}{1+nf(x)} = \frac{\frac{1}{1+x}}{1+\frac{n}{1+x}} = \frac{1}{x+n+1}, n \in \mathbb{N} \cup \{0\}$.
In particular we have $g_0\left(f\left(\frac{1}{2}\right)\right) = \frac{1}{\frac{1}{2}+0+1} = \frac{2}{3}, g_1\left(f\left(\frac{1}{2}\right)\right) = \frac{1}{\frac{1}{2}+1+1} = \frac{2}{5}$.
Also, note that $g_1\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1+1\cdot\frac{1}{2}} = \frac{1}{3}, g_2\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1+2\cdot\frac{1}{2}} = \frac{1}{4}$ and in general
for any $n \geq 3$ we have $g_{n-2}\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1+(n-2)\cdot\frac{1}{2}} = \frac{1}{n}$.
We will prove by MI that any fraction $\frac{m}{n} \in (0,1), m \geq 2$ also can be obtained using only operations $f(x), g(x)$.
Taking fraction $\frac{2}{3}$ as Base of MI and for any natural $n \geq 3$, assuming that any fraction $\frac{p}{q}$ where $2 \leq p < q < n$ can be obtained starting from $\frac{1}{2}$using only operations $f(x), g(x)$, we will prove that any irreducible fraction$\frac{m}{n}$ with $2 \leq m < n$ can be obtained from $\frac{1}{2}$using only operations $f(x), g(x)$ as well.
Indeed, since $\frac{n}{m} = k + \frac{r}{m}, k \geq 1, r \in \{1,\ldots,m-1\}$ ($r \neq 0$ because $\gcd(m,n) = 1$) then $\frac{m}{n} = \frac{1}{\frac{r}{m}+k} = g_{k-1}\left(f\left(\frac{r}{m}\right)\right)$,where $\frac{r}{m}$ by supposition om MI can be obtained using only operations $f(x), g(x)$.

Another way, **using continuous fractions**:
Let $h_n(x) := g_{n-1}(f(x)) = \frac{1}{n+x}, n \in \mathbb{N}$. Also for any $n \in \mathbb{N} \setminus \{1\}$ we have

$$g_{n-2}\left(\frac{1}{2}\right) = \frac{1/2}{1+(n-2)\cdot\frac{1}{2}} = \frac{1}{n}.$$

Let $\frac{a}{b}$ be any fraction in $(0,1)$, that is $1 \leq a < b$.

If $a = 1$ then $\frac{a}{b} = g_{b-2}\left(\frac{1}{2}\right)$;

If $a > 1$ then we have $\frac{a}{b} = [0; n_1, \ldots, n_k] = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ddots + \cfrac{1}{n_k}}} =$

$(h_{n_1} \circ h_{n_2} \circ \ldots \circ h_{n_{k-1}} \circ g_{n_k-2})\left(\frac{1}{2}\right)$.

For examples

$$\frac{13}{29} = \frac{1}{2+\frac{1}{(13/3)}} = \cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{3}}} = h_2\left(h_4\left(g_1\left(\frac{1}{2}\right)\right)\right).$$

or $\frac{5}{13} = g\left(f_3\left(\frac{1}{2}\right)\right)$, that is $\frac{5}{13} = g\left(f\left(f\left(f\left(\frac{1}{2}\right)\right)\right)\right) = (g \circ f \circ f \circ f)\left(\frac{1}{2}\right)$.